SPATIAL TRANSONIC GAS FLOWS IN DUCTS

(O PROSTRANSTVENNYKH TRANSVUKOVYKH TECHENIIAKH GAZA V KANALAKH)

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Frankl' and Falkovich [1,2] have investigated that phase of the laminar flow through a Laval nozzle where a gas accelerates from subsonic to supersonic velocity during the passage through the critical cross-section (throat). Such flows take place when the difference between the pressures at the entrance and at the exit of the nozzle is sufficiently large. If on the other hand, the pressure at the entrance does not exceed very much the pressure on the discharge side, then the stream will be subsonic at both ends of the convergent-divergent nozzle, but it may contain supersonic regions very near the walls adjacent to the region of the critical cross-section. A simple solution of the equations of the gas motion, describing such mixed flow in plane and in axially symmetric nozzles, was given in works [3,4]. In the present note the solution is derived for the analogous spatial flows which contain supersonic regions adjoining the walls of a duct which has two planes of symmetry. By proper choice of the arbitrary constants contained in the solution to be presented, it is possible to increase the regions of local supersonic flow and, as a result, to join them along the axis of the nozzle. In a sense such a flow is singular, because in this case the regime of the flow changes and the velocity field behind the critical cross-section becomes supersonic. Moreover, the solution presented here transforms into the solution obtained in [5], which describes the spatial gas flows in the region of the transition surface from subsonic to supersonic velocities.

We shall investigate the flow of an ideal gas with velocities that differ only infinitesimally from the critical velocity, in a duct which has two planes of symmetry. These two planes intersect along a straight line, namely, the axis of the nozzle. Chosing the x-axis of the cylindrical coordinate system x, r, θ to be coincident with the axis of the nozzle, we shall write the equation which determines the transonic gas flow in the form

$$-\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}=0 \tag{1}$$

where ϕ is the potential of the perturbations. Also

$$\frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial x}=v_x,\qquad \frac{a_*}{\varkappa+1}\frac{\partial\varphi}{\partial r}=v_r,\qquad \frac{a_*}{\varkappa+1}\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}=v_\vartheta \tag{2}$$

where v_x, v_r and v_θ are the perturbations along x, r and θ-axes of a reference velocity, equal in magnitude to the critical velocity a_* and directed along the axis of the nozzle, κ is the Poisson adiabatic coefficient.

Differentiating relationship (1) with respect to x and introducing a new function

$$\frac{\partial\varphi}{\partial x}=u\qquad\left(u=\frac{\varkappa+1}{a_*}v_x\right) \tag{3}$$

we obtain the equation

$$-\frac{1}{2}\frac{\partial^2u^2}{\partial x^2}+\frac{\partial^2u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2u}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=0 \tag{4}$$

To describe the spatial mixed flows which contain local supersonic regions adjoining the walls of the nozzle, we shall look for a solution of equation (4) in the form

$$\frac{\varkappa+1}{a_*}v_x=u=4\frac{d^2}{c^2}(1+k^2+2k\cos 2\vartheta)r^2+4\frac{d}{c^2}g(\xi)$$
$$\xi=cx+d(1+k\cos 2\vartheta)r^2 \tag{5}$$

Substituting expression (5) into equation (4) it is easily seen that the constants c, d, and k may be chosen arbitrarily and that the function $g(\xi)$ must satisfy the ordinary differential equation

$$\frac{d}{d\xi}\left(g\frac{dg}{d\xi}\right)=\frac{dg}{d\xi}+a^2\qquad(a^2=1+k^2) \tag{6}$$

Hereafter we shall assume that everywhere $c>0$ and $d>0$.

Let us find now the remaining velocity components v_r and v_θ. To this end it is simpler to make use of the equations which express the condition of irrotationality of the stream:

$$\frac{1}{r}\frac{\partial v_x}{\partial\vartheta}=\frac{\partial v_\vartheta}{\partial x},\qquad \frac{\partial v_x}{\partial r}=\frac{\partial v_r}{\partial x},\qquad \frac{\partial(rv_\vartheta)}{\partial r}=\frac{\partial v_r}{\partial\vartheta} \tag{7}$$

From the first two equations of this system we have, taking into consideration formulas (5):

$$\frac{\varkappa+1}{a_*}v_\vartheta=-16\frac{d^2}{c^2}kxr\sin 2\vartheta-8\frac{d^2}{c^3}kg(\xi)r\sin 2\vartheta+\frac{1}{r}\chi_1(r,\vartheta)$$
$$\frac{\varkappa+1}{a_*}v_r=8\frac{d^2}{c^2}(1+k^2+2k\cos 2\vartheta)xr+8\frac{d^2}{c^3}(1+k\cos 2\vartheta)g(\xi)r+\chi_2(r,\vartheta) \tag{8}$$

Using the last of the expressions (7), we obtain the first equation which connects the two functions $\chi(r, \theta)$ and $\chi(r, \theta)$ which have to be determined:

$$\frac{\partial \chi_1}{\partial r} = \frac{\partial \chi_2}{\partial \vartheta} \tag{9}$$

The second equation which these functions satisfy may be found by substituting expressions (8) into the equation of motion (1):

$$\frac{\partial \chi_2}{\partial r} + \frac{1}{r^2}\frac{\partial \chi_1}{\partial \vartheta} + \frac{1}{r}\chi_2 = 16\frac{d^3}{c^3}(1+k^2)(1+k\cos 2\vartheta)\, r^2 \tag{10}$$

It follows that the velocity component v_x does not determine uniquely the components v_r and v_θ; on the contrary, to every solution $u(x, r, \theta)$ of equation (4) there corresponds an infinite number of solutions of two linear partial differential equations (9) and (10). The particular solution of these equations is derived from the formulas

$$\chi_1 = r^4 [l \sin 4\vartheta - {}^8/_3\, d^3c^{-3}k\, (1+k^2) \sin 2\vartheta]$$
$$\chi_2 = 4r^3 [-{}^1/_4\, l \cos 4\vartheta + {}^4/_3\, d^3c^{-3}k\, (1+k^2)\cos 2\vartheta + d^3c^{-3}(1+k^2)] \tag{11}$$

The remaining solutions of the system (9) and (10) may be expressed, using the principle of superposition, in terms of harmonic functions. To describe the flows in the nozzles, the cross-section of which has two axes of symmetry, it is sufficient to make use of the solution (11), because then the functions v_x and v_r will be even with respect to θ while the function v_θ will be odd.

Now let us investigate the function $g(\xi)$ in greater detail. To this end we integrate equation (6) and obtain as a result

$$g\,\frac{dg}{d\xi} = g + a^2\xi \tag{12}$$

The constant of integration here is included in ξ, since equations (1) and (4) are invariant with respect to a displacement along the x-axis. The differential equation of the first order (12) has one singular point $\xi = g = 0$, which for any value of a is a saddle point. On the curve $g = -a^2\xi$ the derivative becomes zero, for $g = 0$ this derivative is infinite. Therefore, the qualitative configuration of the integral curves of equation (12) will be the same for any values of the parameter a, as is shown in Fig. 1. To obtain an exact solution of this equation we shall introduce the change of variables

$$\frac{dg}{d\xi} = q \tag{13}$$

From the resulting equation (6) we have now

$$gq\,\frac{dq}{dg} = q - q^2 + a^2 \tag{14}$$

Equation (14) is easily integrable:

$$g = [e\,(1-q_1)^{-q_1}(1-q_2)^{q_2}(q-q_1)^{q_1}(q-q_2)^{-q_2}]^{\frac{1}{q_2-q_1}} \tag{15}$$

where e is an arbitrary constant which may be real or complex, the quantities q_1 and q_2 are expressed in terms of a^2 by means of equations

$$q_1 = {}^1/_2\,(1-\sqrt{1+4a^2}), \qquad q_2 = {}^1/_2\,(1+\sqrt{1+4a^2}) \tag{16}$$

Eliminating $dg/d\xi = q$ from relationships (12) and (15) we obtain the formula which defines the function $g(\xi)$ in implicit form:

$$(g + a^2\xi\,/\,q_1)^{q_2}\,(g + a^2\xi\,/\,q_2)^{-q_1} = e \tag{17}$$

For $k = 1$, i.e. in the case laminar flows, expression (17) becomes the solution obtained in [3]; equation (17) describes the flows with axial symmetry investigated in [4].

For the description of the mixed flows we have to choose from the four families of curves, shown in Fig. 1, the curves of the family A. Indeed, in the solutions which correspond to the branches B and B^1, the derivatives of the velocity components of the stream become infinite at the points $g = 0$; if we chose branch A^1 for the desired solution $g(\xi)$, then the velocity field in the duct will be supersonic. The dimensions of the local supersonic zone depends on the magnitude of the constant e which appears in formula (17). When e is decreased the region of the supersonic velocities is increased and for $e = 0$, the supersonic zone is extended to the axis of the nozzle. In this limiting case the graph of the function $g(\xi)$ is represented by the broken curve aob. Then the type of the flow changes, because the velocity field downstream of the critical cross-section becomes supersonic. The corresponding solution $g(\xi)$ is represented in Fig. 1 by the straight line aoc; function v_x in terms of Cartesian coordinates $z = r\cos\theta$, $y = r\sin\theta$ is given by the formula obtained by the author in [5]:

$$\begin{aligned}\frac{\varkappa+1}{a_*}v_x &= 2\,\frac{d}{c}\,(1 \mp \sqrt{1+4a^2})\,x + \\ &+ 2\,\frac{d^2}{c^2}\,[3 + 2k^2 \mp \sqrt{1+4a^2} + (5 \mp \sqrt{1+4a^2})\,k]\,z^2 + \\ &+ 2\,\frac{d^2}{c^2}\,[3 + 2k^2 \mp \sqrt{1+4a^2} - (5 \mp \sqrt{1+4a^2})\,k]\,y^2\end{aligned} \tag{18}$$

The form of the function v_r and v_θ is easily obtainable when using equations (8) and (11).

The derived solutions as given by formulas (5) and (8) may be used to describe the gas motion, as it is varied systematically by increasing the difference of pressure between the entrance and the exit of the nozzle. Also the extent of the supersonic zone, which is formed initially near the walls of the channel in the region of the critical cross-section, is increased gradually. When the difference of pressure is further increased, the surface of transition from subsonic to supersonic flow develops in

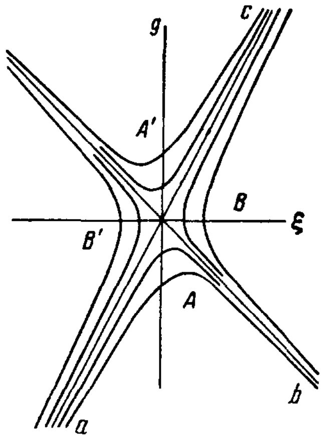

Fig. 1.

the region of the critical cross-section. The gas flow in this region is described by formula (18). The solution which is given by the equation

$$e = 0 \qquad (g = -a^2\xi / q_{1,2}) \tag{19}$$

which describes a singular flow with a local supersonic zone is not an analytical solution. The derivatives of the velocity components, which correspond to it, contain a discontinuity on the surface:

$$x = -dc^{-1}(1 + k\cos 2\vartheta)\, r^2 \tag{20}$$

This surface, according to the theory of partial differential equations, is characteristic for equation (4). A detailed investigation of solutions with second derivatives of the velocity potential, having discontinuities on the particular characteristic surfaces, was carried out in [6].

We shall introduce the notations

$$2\,\frac{d}{c}\,(1 \mp \sqrt{1 + 4a^2}) = A, \qquad -\frac{1}{4}\;\frac{5 \mp \sqrt{1 + 4a^2}}{3 + 2k^2 \mp \sqrt{1 + 4a^2}}\,k = n \tag{21}$$

Then the expressions (18) will have the form [5, 6]:

$$\frac{\varkappa + 1}{a_*}\,v_x = A_x + A^2\,(^1/_4 - n)\,z^2 + A^2\,(^1/_4 + n)\,y^2 \tag{22}$$

From formulas (20) and (22) it follows that quantity k characterizes the degree of deviation of the form of the particular characteristic surface from a surface of revolution, and parameter n characterizes the degree of deviation of the stream velocity field from axial symmetry. It is interesting to point out that one of the two solutions, given by the equation $e = 0$, while describing the gas motion with a supersonic velocity field downstream of the critical cross-section of the nozzle does not account for all possible flows of that type. Only the flows to which correspond values of n of absolute magnitude not exceeding 5/16 are represented by formula (18). This is the case, because in the solutions developed in this paper in accordance with relationship (20), the characteristic surfaces always pass through the center of the nozzle. At the same time in [5] it has been shown that similar surfaces exist only for n lying in the interval $-5/16 \leqslant n \leqslant 5/16$. The absence of particular characteristic surfaces in the solutions for $|n| > 5/16$ and also the fact that such solutions cannot be obtained from formula (5) by means of a limiting process, when the magnitude of constant e is approaching zero, point to the instability of the corresponding mixed gas flows.

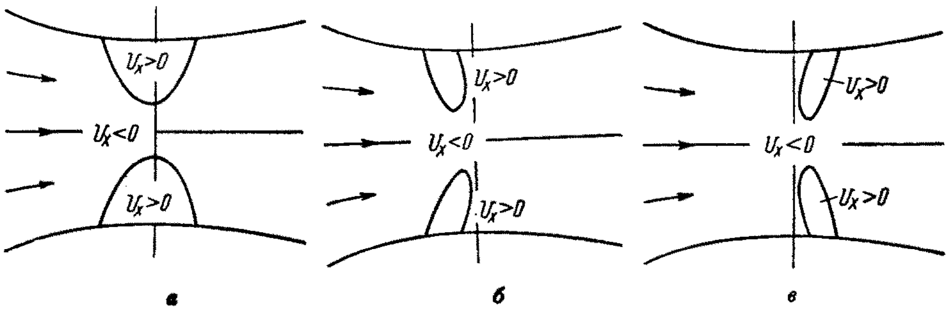

Fig. 2.

In those cases when values e are near zero, the graph of function $g(\xi)$ does not differ greatly from the broken curve aob shown in Fig. 1. Therefore, using formula (18), it is easy to obtain the shape of the curves which are formed by the intersection of the sonic surface with the mutually perpendicular planes $y = 0$ and $z = 0$. It may be shown that for $|k| < 1/3$ in both of these planes, curves are obtained which have the usual form; they are represented in Fig. 2a. The curves shown in Fig. 2b,c are obtained for $|k| > 1/3$ in either the plane $y = 0$ or $z = 0$. The first of these figures corresponds to the case $1/3 < |k| < 1$ and the second to $-|k| > 1$. The form of the intersection of the sonic surface with the second plane $z = 0$ or $y = 0$, respectively for these values of k are qualitatively the same as for $|k| < 1/3$. In Figs. 2a, b, c the abscissa coincides with one of the axes y or z depending upon the sign of k.

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